Fundamentals of collapsing (or condensation)
of multi-group cross section \(^1\)

Go CHIBA

1 Infinite homogeneous systems

Let us consider a infinite homogeneous system including external neutron sources and no fissile materials. The original (pre-collapsing) multi-group neutron slowing-down equation can be presented as follows:

\[
\Sigma_{a,g} \phi_g + \sum_{g'} \Sigma_{g' \to g} \phi_{g'} = \sum_{g'} \Sigma_{g' \to g} \phi_{g'} + s_g, \tag{1}
\]

where \(s_g\) is an external neutron source in group \(g\).

Next let us consider macro-group \(G\), which contains some energy groups in the original energy group structure, and take sums over this macro-group in the both sides of the above equation. Then the following equation can be derived:

\[
\sum_{g \in G} \Sigma_{a,g} \phi_g + \sum_{g \in G} \sum_{g' \in G'} \Sigma_{g' \to g} \phi_{g'} = \sum_{g' \in G'} \sum_{g \in G} \Sigma_{g' \to g} \phi_{g'} + \sum_{g \in G} s_g. \tag{2}
\]

Now let us define \(\tilde{\phi}_G\) and \(\tilde{s}_G\) as

\[
\tilde{\phi}_G = \sum_{g \in G} \phi_g, \tag{3}
\]
\[
\tilde{s}_G = \sum_{g \in G} s_g. \tag{4}
\]

By using them, Eq. (2) can be rewritten as

\[
\sum_{g \in G} \Sigma_{a,g} \phi_g \tilde{\phi}_G + \sum_{g \in G} \sum_{g' \in G'} \Sigma_{g' \to g} \phi_{g'} \tilde{\phi}_G = \sum_{g' \in G'} \sum_{g \in G} \Sigma_{g' \to g} \phi_{g'} \tilde{\phi}_G + \tilde{s}_G. \tag{5}
\]

Multi-group cross sections in the macro-group structure, which are post-collapsing cross sections, are defined as follows:

\[
\tilde{\Sigma}_{a,G} = \frac{\sum_{g \in G} \Sigma_{a,g} \phi_g}{\tilde{\phi}_G}, \tag{6}
\]
\[
\tilde{\Sigma}_{G' \to G} = \frac{\sum_{g \in G} \sum_{g' \in G'} \Sigma_{g' \to g} \phi_{g'}}{\tilde{\phi}_G}. \tag{7}
\]

These correspond to cross sections collapsed with neutron flux weighting. By using these collapsed cross sections, Eq. (5) can be written as

\[
\tilde{\Sigma}_{a,G} \tilde{\phi}_G + \sum_{G'} \tilde{\Sigma}_{G' \to G} \tilde{\phi}_G = \sum_{G'} \tilde{\Sigma}_{G' \to G} \tilde{\phi}_G + \tilde{s}_G. \tag{8}
\]

This equation suggests that neutron flux obtained by solving neutron slowing-down equation with collapsed cross sections, \(\tilde{\phi}_G\), becomes identical with \(\sum_{g \in G} \phi_g\), that is, integrated neutron flux can be preserved through cross section collapsing, if cross section collapsing is done with neutron flux weighting and external neutron source in collapsed group structure is defined by Eq. (4). Also integrated reaction rate defined as a product of multi-group cross section and neutron flux is also preserved.

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Next let us discuss a system which includes fissile materials and consider neutron multiplication factors before and after cross section collapsing. In this case, external neutron source is replaced by fission neutron source. If total fission neutron source is normalized to unity, multi-group fission neutron source becomes fission spectrum $\chi_g$. If we define collapsed fission spectrum $\tilde{\chi}_G$ as

$$\tilde{\chi}_G = \sum_{g \in G} \chi_g,$$

(9)

and multi-group cross sections are collapsed with neutron flux, obtained by solving the original slowing-down equation, as weight, integrated neutron flux and reaction rate can be preserved through cross section collapsing. Neutron multiplication factor in the original problem $k$ can be defined with total neutron production and absorption rates as

$$k = \frac{\sum \nu \Sigma_{f,g} \phi_g}{\sum \Sigma_{a,g} \phi_g}. \quad (10)$$

If integrated reaction rates are preserved through group collapsing, the following equation can be derived:

$$k = \frac{\sum \nu \Sigma_{f,G} \tilde{\phi}_G}{\sum \Sigma_{a,G} \tilde{\phi}_G} = \tilde{k}. \quad (11)$$

This equation suggests that neutron multiplication factor in a collapsed system $\tilde{k}$ is identical with $k$.

2 Infinite homogeneous system with buckling

Next let us consider infinite homogeneous systems in which neutron leakage is considered by the buckling. If the total fission source is normalized to unity, the homogeneous B1 equation, which describes one-point neutron slowing-down problems with buckling, can be written as

$$\Sigma^0_{t,g}\phi_g + iB J_g = \sum_{g'} \Sigma^0_{g' \rightarrow g} \phi_{g'} + \chi_g, \quad (12)$$

$$iB \phi_g + 3\alpha_g \Sigma^1_{t,g} J_g = 3\sum_{g'} \Sigma^1_{g' \rightarrow g} J_{g'}, \quad (13)$$

where $\Sigma^0_{g' \rightarrow g}$ is the $n$th-order Legendre moment of scattering cross sections. Both of $\Sigma^0_{t,g}$ and $\Sigma^1_{t,g}$ are total cross sections, but these are cross sections collapsed with different weights, so this difference in the weight is distinguished by the superscripts. $iB J_g$ is neutron current, and $\alpha_g$ is defined as

$$\alpha_g = \frac{1}{3} \left( \frac{B}{\Sigma^0_{t,g}} \right)^2 \frac{A_g \Sigma^0_{t,g}}{1 - A_g \Sigma^0_{t,g}}, \quad (14)$$

$$A_g = \frac{1}{2} \int_{-1}^{1} \frac{1}{\Sigma^0_{t,g} + iB \mu} d\mu, \quad (15)$$

and $A_g$ is explicitly presented as

$$A_g = \frac{1}{B} \tan^{-1} \left( \frac{B}{\Sigma^0_{t,g}} \right) (B^2 > 0), \quad (16)$$

$$A_g = \frac{1}{2k} \ln \left( \frac{1 + \frac{k}{\Sigma^0_{t,g}}}{1 - \frac{k}{\Sigma^0_{t,g}}} \right) (k^2 = -B^2 > 0). \quad (17)$$

If $\alpha_g$ is set to unity, this becomes the homogeneous P1 equation.

Here let us discuss group collapsing of cross sections in Eqs. (12) and (13). Neutron flux should be used as weight for $\Sigma^0_{t,g}$ and $\Sigma^0_{g' \rightarrow g}$, and neutron current should be used for $\Sigma^1_{g' \rightarrow g}$. $\alpha$ in collapsed group structure should not be calculated
from collapsed cross section $\Sigma^0_{l,G}$ with Eq. (14), but $\tilde{\alpha}_G \tilde{\Sigma}^1_{l,G}$ should be calculated as

$$\tilde{\alpha}_G \tilde{\Sigma}^1_{l,G} = \frac{\sum_{g \in G} \alpha_g \Sigma^1_{l,g} J_g}{\sum_{g \in G} J_g}.$$  \hspace{1cm} (18)

If we do so, neutron flux, neutron current, product of multi-group cross section and these quantities and neutron multiplication factor can be preserved through group collapsing.

### 3 Heterogeneous systems

In heterogeneous systems, a specific term, the diffusion term in the diffusion equation or the collision term in the transport equation, should be taken into account in group collapsing, so integrated quantities cannot be preserved even if multi-group cross sections are collapsed with proper weight functions such as neutron flux and neutron current.[1, 2]

As an example, let us consider the one-dimensional neutron transport equation. When angular neutron flux in group $g$ is denoted as $\psi_g(\mu)$, the collision term can be written as $\Sigma^1_{t,g} \psi_g(\mu)$. Note that $\mu$ is the cosine between neutron direction and the X-axis. When we want to preserve integrated collision term, we have to define collapsed cross section as

$$\tilde{\Sigma}_{l,G}(\mu) = \frac{\sum_{g \in G} \Sigma_{t,g} \psi_g(\mu)}{\sum_{g \in G} \psi_g(\mu)}.$$  \hspace{1cm} (19)

and this means that collapsed cross sections should be dependent on neutron direction $\mu$. To avoid this problem, angular neutron flux in the original problem is expanded by the Legendre polynomial. In this case, the collision term can be presented as $\Sigma_{t,g} \sum_l \frac{2l+1}{2} P_l(\mu) \phi_{l,g}$, so collapsed cross sections which should depend on the Legendre-moment can be defined as

$$\tilde{\Sigma}^l_{t,G} = \frac{\sum_{g \in G} \Sigma_{t,g} \phi_{l,g}}{\sum_{g \in G} \phi_{l,g}}.$$  \hspace{1cm} (20)

In this case, the collision term in collapsed system can be transformed to

$$\sum_l \frac{2l+1}{2} P_l(\mu) \tilde{\Sigma}^l_{t,G} \phi_{l,G} = \sum_l \frac{2l+1}{2} P_l(\mu) \left( \tilde{\Sigma}^1_{t,G} + \tilde{\Sigma}^l_{t,G} \right) \phi_{l,G}$$

$$= \tilde{\Sigma}^1_{t,G} \sum_l \frac{2l+1}{2} P_l(\mu) \phi_{l,G} + \sum_l \frac{2l+1}{2} P_l(\mu) \left( -\tilde{\Sigma}^1_{t,G} + \tilde{\Sigma}^l_{t,G} \right) \phi_{l,G}$$

$$= \tilde{\Sigma}^1_{t,G} \psi_G(\mu) + \sum_l \frac{2l+1}{2} P_l(\mu) \left( -\tilde{\Sigma}^1_{t,G} + \tilde{\Sigma}^l_{t,G} \right) \phi_{l,G}.$$  \hspace{1cm} (21)

By moving the second term in the third line of the above equations to the scattering term, the conventional form of the neutron transport equation can be derived.[3].

### References

